On a phase transition of the random intersection graph: supercritical region

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This is joint work with Jeong Han Kim and Joohan Na (KIAS).

Motivation

Question

What was the most issued word in Korea this year?

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Multiple choices:

1. Gangnam style
2. ICM Seoul 2014
3. Mers (Middle East Respiratory Syndrome)

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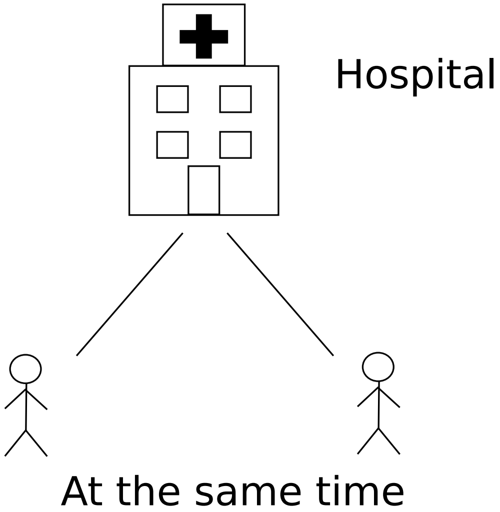
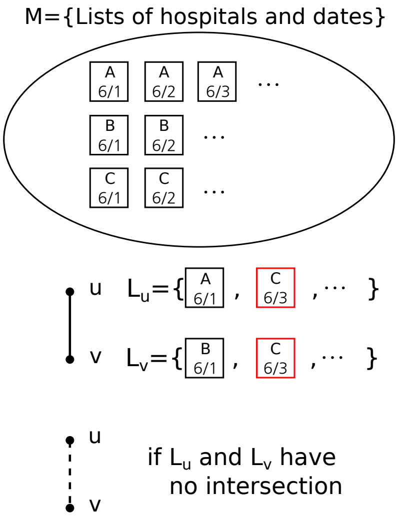
What is the proper network (or graph) model explaining epidemic of Mers?

Answer: Random intersection graph

Feature of Mers in Korea



Graph model about epidemic of Mers



Sec 1) Definition

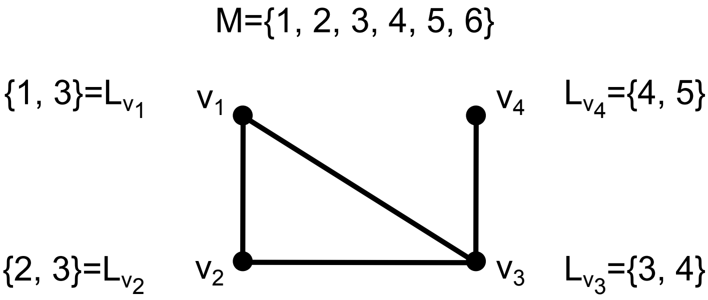
- $V := \{v_1, \dots, v_n\}$
- $\{L_1, \dots, L_n\}$: a collection of sets

Definition (Intersection graph)

The *intersection graph* on V generated by $\{L_1, \dots, L_n\}$ is the graph on V in which

$$v_i \sim v_j \quad \text{if and only if} \quad L_i \cap L_j \neq \emptyset.$$

Example



Random intersection graph

Definition (Random Intersection graph $G(n, m; p)$)

- M: a set of size m.
- L_i: a random subset obtained by choosing each element in M indepedently with probabiity p.

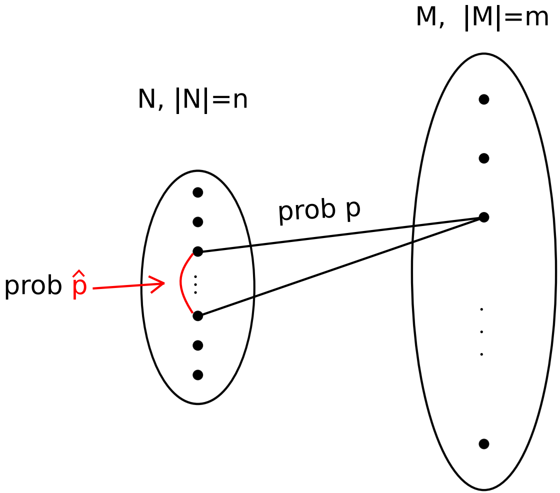
Random intersection graph

Definition (Random Intersection graph $G(n, m; p)$)

- M: a set of size m.
- L_i: a random subset obtained by choosing each element in M indepedently with probabiity p.
- The *random intersection graph* $G(n, m; p)$ is the intersection graph generated by i.i.d. L_i as above.

It was defined by Karoński, Scheinerman, and Singer-Cohen (1999).

Visualization: Random bipartite graph



Application

1. A random intersection graph has received a lot of attention because of a great diversity of applications:
    - Epidemic
    - Circuit design
    - Network user profiling
    - Analysis of complex networks.

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1. A random intersection graph has received a lot of attention because of a great diversity of applications:
   - Epidemic
   - Circuit design
   - Network user profiling
   - Analysis of complex networks.
2. The special case when L_i's are uniformly distributed as subsets of M of the same size has been applied to security of wireless sensor networks.

Question

When is $G(n, m; p)$ essentially the same as the binomial random graph $G(n, \hat{p})$ with the same expected number of edges?

Remark: $\hat{p} := 1 - (1 - p^2)^m \sim mp^2$ if mp^2 is small.

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Distance between two random graphs: Total variation.

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Notion

Distance between two random graphs: Total variation.

Definition

The total variation between two random graphs X and Y is defined by

$$\mathrm{TV}(X, Y) := \frac{1}{2} \sum_{G} \Big| \Pr[X = G] - \Pr[Y = G] \Big|,$$

where the sum is over all possible graphs G of X and Y.

Sec 2) Previous results

Observation

Let $\omega \to \infty$ as $n \to \infty$.

1. If $p \leq \frac{1}{\omega n \sqrt{m}}$,
   then two random graphs are the empty graph with high probability.

Sec 2) Previous results

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Let $\omega \to \infty$ as $n \to \infty$.

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   with high probability.
2. If $p \geq \sqrt{\frac{2 \ln n + \omega}{m}}$,
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   then two random graphs are the complete graph with high probability.

Assumption

$$\frac{1}{\omega n \sqrt{m}} \leq p \leq \sqrt{\frac{2 \ln n + \omega}{m}}.$$

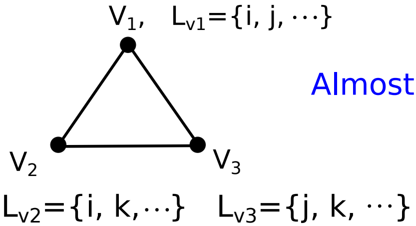
Proposition

If $m \ll n^3$ and $\frac{\omega}{n\sqrt{m}} \leq p \leq \sqrt{\frac{2\ln n - \omega}{m}}$, then

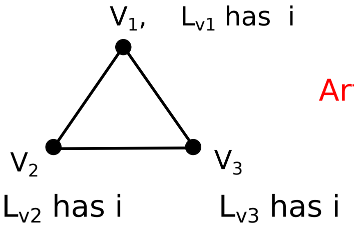
$$\mathrm{TV}\Big(G(n,m;p), G(n,\hat{p})\Big) \to 1.$$

Idea: By comparing the number of triangles.

Two types of triangles in $G(n, m; p)$



Almost independent



Artifact triangle

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Proof:

1. X := the number of independent triangles
   Y := the number of artifact triangles
2. $tr(G(n,m;p)) = X + Y$ and $tr(G(n,\hat{p})) = X$.
3. With high probability,

$$E[X+Y] + \omega(\sigma(X)) \leq tr(G(n,m;p)) \leq E[X+Y] + \omega(\sigma(X))$$

$$E[X] + \omega(\sigma(X)) \leq tr(G(n,\hat{p})) \leq E[X] + \omega(\sigma(X))$$

4. If $m \ll n^3$, then $\sigma(X) \ll E[Y]$.
5. $tr(G(n,m;p)) \gg tr(G(n,\hat{p}))$ with high probability.

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Theorem (Fill, Scheinerman and Singer-Cohen (2000))

If $m = n^{\alpha}$ and $\alpha > 6$, then

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If $m = n^\alpha$ *and* $3 < \alpha \leq 6$, *for any monotone property* $\mathcal{P}$, $\Pr[G(n, m; p) \in \mathcal{P}]$ *is similar to* $\Pr[G(n, \hat{p}) \in \mathcal{P}]$ *(with a technical statement).*

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Question

What is the total variation if $n^3 \ll m \ll n^6$?

Problem

What is the smallest constant α such that for $m = n^{\alpha}$ and any $p = p(n)$,

$$\mathrm{TV}\Big(G(n, m; p), G(n, \hat{p})\Big) = o(1)?$$

Previous result

$$3 \leq \alpha \leq 6.$$

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for $m = n^{\alpha}$ and any $p = p(n)$,

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Previous result

$$3 \leq \alpha \leq 6.$$

Main Theorem (Kim, Lee, Na (2015+))

For $m \gg n^4$ and $0 \leq p = p(n) \leq 1$,

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In Progress

If $m = \frac{n^4}{\log \log n}$, then for $p = \frac{c}{\sqrt{m}}$,

$$\mathrm{TV}\Big(G(n, m; p), G(n, \hat{p}) \Big) \geq \frac{1}{2}.$$

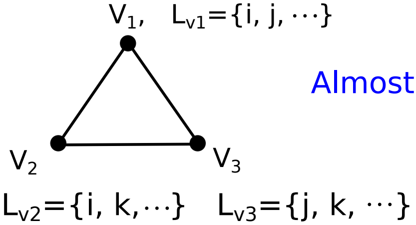
* We believe that 4 in the exponent is tight.

Artifact triangles

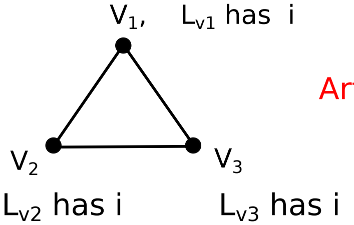
Recall

An artifact triangle is a triangle formed by the same element in M.

1. Fill, Scheinerman and Singer-Cohen (2000):
   the case when there is no artifact triangle.
2. Kim, Lee and Na (2015+):
   the case when there are not so many artifact triangles.



Almost independent



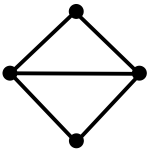
Artifact triangle

Key object

Key object: Diamond graph

- A diamond graph $= K_4$ minus one edge.
- The number of diamond graphs with two artifact triangles in $G(n, m; p)$ is small iff

$$G(n, m; p) \sim G(n, \hat{p}).$$



Diamond graph

Sec 3) Outline of Proof of Main Theorem

Recall: Main Theorem (Kim, Lee, Na (2015+))

For $m \gg n^4$ and $0 \le p \le 1$,

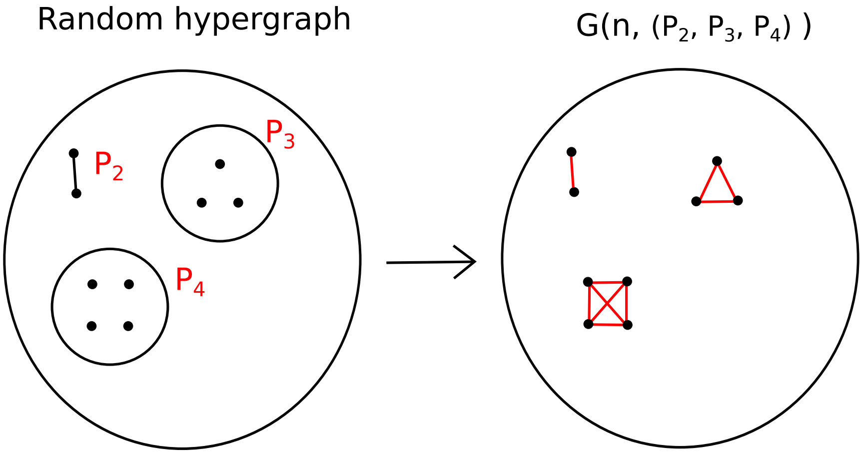
$$\mathrm{TV}\Big(G(n,m;p),\, G(n,\hat{p})\Big) = o(1).$$

Remark (essentially by Rybarczyk)

$G(n,m;p)$ is approximated by a random graph $G(n,(p_2,p_3,p_4))$, where

$$p_k := 1 - e^{-mp^k(1-p)^{n-k}}.$$

Definition of $G(n,(p_2,p_3,p_4))$



Remark: Why $p_k := 1 - e^{-mp^k(1-p)^{n-k}}$?

- For $a \in M$, $V_a := \{v : a \in L_v\}$.
- For a fixed k-subset $U \subset V$,

$$\Pr\left[\exists a \in M \text{ s.t. } V_a = U\right] = 1 - (1 - p^k(1-p)^{n-k})^m.$$

Recall: Main Theorem (Kim, Lee, Na (2015+))

For $m \gg n^4$ and $0 \leq p \leq 1$,

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Key Lemma (Kim, Lee, Na (2015+))

For $m \gg n^4$ and $0 \leq p \leq (\frac{3\log n}{m})^{1/2}$,

$$\mathrm{TV}\Big(G(n,(p_2,p_3,p_4)), G(n,p_2)\Big) = o(1).$$

Sec 4) Proof of Lemma

$$\mathrm{TV}\Big(G(n,(p_2,p_3,p_4)),G(n,p_2)\Big)$$
$$:= \frac{1}{2}\sum_{G}\Big|\Pr[X=G]-\Pr[Y=G]\Big|$$
$$= \sum_{G\in\mathcal{G}}\Big(\Pr[G(n,p_2)=G]-\min\Big\{\Pr[G(n,(p_2,p_3,p_4))=G],\Pr[G(n,p_2)=G]\Big\}\Big).$$

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$$= \sum_{G \in \mathcal{G}} \Big(\Pr[G(n,p_2) = G] - \min\Big\{ \Pr[G(n,(p_2,p_3,p_4)) = G], \Pr[G(n,p_2) = G] \Big\} \Big).$$

Observation

If $\Pr[G(n,(p_2,p_3,p_4)) = G] \geq (1 - O(\epsilon))\Pr[G(n,p_2) = G]$, then

$$\mathrm{TV}\Big(G(n,(p_2,p_3,p_4)),\, G(n,p_2)\Big) = O(\epsilon).$$

$$\Pr[G(n,(p_2,p_3,p_4)) = G]$$

$$= \sum_{\substack{Q\subseteq \mathcal{H}_4(G)\\ T\subseteq \mathcal{H}_3(G)}} \Pr[\mathcal{H}_4(n,p_4) = Q, \mathcal{H}_3(n,p_3) = T, G(n,p_2) = G]$$

$$= \sum_{\substack{Q\subseteq \mathcal{H}_4(G)\\ T\subseteq \mathcal{H}_3(G)}} p_4^{|Q|}(1-p_4)^{\binom{n}{4}-|Q|} p_3^{|T|}(1-p_3)^{\binom{n}{3}-|T|} p_2^{|G|-|K(Q)\cup K(T)|}(1-p_2)^{\binom{n}{2}-|G|}$$

$$= \Pr[G(n,p_2) = G] \sum_{\substack{Q\subseteq \mathcal{H}_4(G)\\ T\subseteq \mathcal{H}_3(G)}} p_4^{|Q|}(1-p_4)^{\binom{n}{4}-|Q|} p_3^{|T|}(1-p_3)^{\binom{n}{3}-|T|} p_2^{-|K(Q)\cup K(T)|}.$$

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Claim

$$\frac{\Pr[G(n,(p_2,p_3,p_4)) = G]}{\Pr[G(n,p_2) = G]} \geq \sum_{Q \subseteq \mathcal{H}_4(G)} p_4^{|Q|}(1-p_4)^{\binom{n}{4}-|Q|} p_2^{-|K(Q)|}$$
$$\times \sum_{T \subseteq \mathcal{H}_3(G \setminus K(Q))} p_3^{|T|}(1-p_3)^{\binom{n}{3}-|T|} p_2^{-|K(T)|}.$$

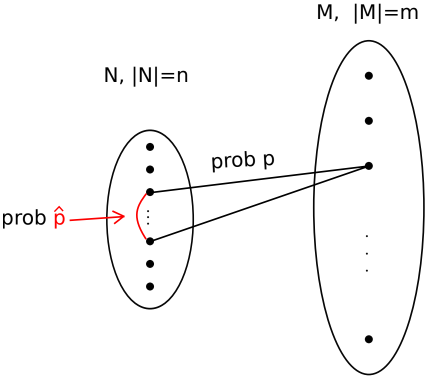
Problem

Problem

Fix $3 < \alpha < 4$, and let $m = n^{\alpha}$.
Find a probability $p^* = p^*(n, m)$ such that

- If $p \ll p^*$, then $\mathrm{TV}\left(G(n, m; p), G(n, \hat{p})\right) = o(1)$.
- If $p \gg p^*$, then $\mathrm{TV}\left(G(n, m; p), G(n, \hat{p})\right) \geq c$,
  for some positive constant $c > 0$.

Problem



Problem

1. non-uniform version with p_{ij}
2. The red edge is exposed with a probability q.

Thank you for your attention!

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Three cases

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- Case I: no artifact triangles
- Case II: $\exists$ artifact triangles and no artifact quadruples
- Case III: $\exists$ artifact quadruples

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$$\mathrm{TV}\Big(G(n,(p_2,p_3,p_4)), G(n,p_2)\Big) \le O(\varepsilon).$$

Remark

It gives the result by Fill–Scheinerman–Singer-Cohen (2000).

Case II: ∃ artifact triangles and no artifact quadruples

In this case, the expected number of artifact triangles is not small, but the expected number of artifact quadruples is small, that is,

$$\frac{\varepsilon}{nm^{1/3}} < p \le \frac{\varepsilon}{n^{2/3}m^{1/3}}.$$

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Remark: It is not possible for an arbitrary G to show

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Idea: We consider properties of typical $G \in G(n, \hat{p})$.

For any family $\mathcal{G}_3$ of typical graphs on V,

$$\mathrm{TV}\Big(G(n,(p_2,p_3,p_4)),G(n,p_2)\Big)$$
$$\le \Pr[G(n,p_2)\notin\mathcal{G}_3]$$
$$+\sum_{G\in\mathcal{G}_3}\Big(\Pr[G(n,p_2)=G]-\min\Big\{\Pr[G(n,(p_2,p_3,p_4))=G],\Pr[G(n,p_2)=G]\Big\}\Big)$$
$$\le O(\varepsilon)+\sum_{G\in\mathcal{G}_3}\Big(\Pr[G(n,p_2)=G]-\min\Big\{\Pr[G(n,(p_2,p_3,p_4))=G],\Pr[G(n,p_2)=G]\Big\}\Big).$$

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&\quad + \sum_{G\in\mathcal{G}_3} \Big(\Pr[G(n,p_2){=}G]-\min\big\{\Pr[G(n,(p_2,p_3,p_4)){=}G],\Pr[G(n,p_2){=}G]\big\}\Big)\\
&\le O(\varepsilon) + \sum_{G\in\mathcal{G}_3} \Big(\Pr[G(n,p_2){=}G]-\min\big\{\Pr[G(n,(p_2,p_3,p_4)){=}G],\Pr[G(n,p_2){=}G]\big\}\Big).
\end{aligned}$$

Goal

For any typical $G \in \mathcal{G}_3$,

$$\Pr[G(n,(p_2,p_3,p_4)) = G] \ge (1 - O(\varepsilon))\Pr[G(n,p_2) = G].$$

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- $I(G)$: the number of diamond graphs, i.e., K_4 minus one edge.

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Lemma

Let $\mathcal{G}_3$ be the set of all graphs G on V satisfying

$$|\mathcal{H}_3(G)| \geq (1-\delta)\binom{n}{3}p_2^3 \quad \text{and} \quad I(G) \leq n^4 p_2^5/\varepsilon.$$

Then, for $\frac{\varepsilon}{nm^{1/3}} < p \leq \left(\frac{3\log n}{m}\right)^{1/2}$,

$$\Pr[G(n,p_2) \in \mathcal{G}_3] = 1 - O(\varepsilon).$$

Taking $Q = \emptyset$, we have

$$\frac{\Pr[G(n,(p_2,p_3,p_4)) = G]}{\Pr[G(n,p_2) = G]} \geq (1-p_4)^{\binom{n}{4}} \sum_{T \subseteq \mathcal{H}_3(G)} p_3^{|T|}(1-p_3)^{\binom{n}{3}-|T|} p_2^{-|K(T)|}$$

$$\geq (1-O(\varepsilon)) \sum_{t=0}^{t_0} \sum_{\substack{T \subseteq \mathcal{H}_3(G) \\ |T|=t, |K(T)|=3t}} p_3^t (1-p_3)^{\binom{n}{3}-t} p_2^{-3t},$$

where

$$t_0 := \frac{n^3 m p^3}{\varepsilon} = \Theta\left(\frac{n^3 p_3}{\varepsilon}\right).$$

$$\frac{\Pr[G(n,(p_2,p_3,p_4)) = G]}{\Pr[G(n,p_2) = G]}$$

$$\geq (1-O(\varepsilon)) \sum_{t=0}^{t_0} \sum_{\substack{T \subseteq \mathcal{H}_3(G) \\ |T|=t, |K(T)|=3t}} p_3^t (1-p_3)^{\binom{n}{3}-t} p_2^{-3t}$$

$$= (1-O(\varepsilon)) \sum_{t=0}^{t_0} (1-O(\varepsilon)) \binom{\binom{n}{3}}{t} p_2^{3t} p_3^t (1-p_3)^{\binom{n}{3}-t} p_2^{-3t}$$

$$\geq (1-O(\varepsilon)) \sum_{t=0}^{t_0} \binom{\binom{n}{3}}{t} p_3^t (1-p_3)^{\binom{n}{3}-t}$$

$$= (1-O(\varepsilon)) \left(1 - \Pr\left[\mathrm{Bin}\left(\binom{n}{3}, p_3\right) > t_0\right]\right) = 1 - O(\varepsilon).$$

It implies that

$$\mathrm{TV}\Big(G(n,(p_2,p_3,p_4)), G(n,p_2)\Big) = O(\varepsilon).$$

Case III: $\exists$ artifact triangles and quadruples

In this case, the expected number of artifact quadruples is not small, that is,

$$\frac{\varepsilon}{n^{2/3}m^{1/3}} < p \leq \left(\frac{3\log n}{m}\right)^{1/2}.$$

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- $|\mathcal{H}_4(G)|$: the number of quadruples in G.

Lemma

Let $\mathcal{G}_4 \subset \mathcal{G}_3$ be the set of all graphs G on V satisfying

$$|\mathcal{H}_4(G)| \geq \left(1 - \frac{1}{\varepsilon n}\right)\binom{n}{4}p_2^6.$$

Then, for $\frac{\varepsilon}{n^{2/3}m^{1/3}} < p \leq \left(\frac{3\log n}{m}\right)^{1/2}$,

$$\Pr[G(n, p_2) \in \mathcal{G}_4] = 1 - O(\varepsilon).$$

Goal

For any $G \in \mathcal{G}_4$,

$$\Pr[G(n,(p_2,p_3,p_4)) = G] \geq (1 - O(\varepsilon)) \Pr[G(n,p_2) = G].$$

Goal

For any $G \in \mathcal{G}_4$,

$$\Pr[G(n,(p_2,p_3,p_4)) = G] \geq (1 - O(\varepsilon)) \Pr[G(n,p_2) = G].$$

$$\frac{\Pr[G(n,(p_2,p_3,p_4)) = G]}{\Pr[G(n,p_2) = G]}$$

$$\geq \sum_{Q \subseteq \mathcal{H}_4(G)} p_4^{|Q|}(1-p_4)^{\binom{n}{4}-|Q|} p_2^{-|K(Q)|} \cdot \sum_{T \subseteq \mathcal{H}_3(G \setminus K(Q))} p_3^{|T|}(1-p_3)^{\binom{n}{3}-|T|} p_2^{-|K(T)|}$$

$$\geq (1 - O(\varepsilon)) \cdot \min_{\substack{Q \subseteq \mathcal{H}_4(G) \\ |Q| \leq q_0}} \sum_{T \subseteq \mathcal{H}_3(G \setminus K(Q))} p_3^{|T|}(1-p_3)^{\binom{n}{3}-|T|} p_2^{-|K(T)|},$$

where $q_0 := \frac{n^4 m p^4}{\varepsilon} = \Theta\left(\frac{n^4 p_4}{\varepsilon}\right)$.

Let $t_0 := \dfrac{n^3 m p^3}{\varepsilon} = \Theta\left(\dfrac{n^3 p_3}{\varepsilon}\right)$ and $r := \dfrac{n^4 m^2 p^6}{\varepsilon^3} = \Theta\left(\dfrac{n^4 p_3^2}{\varepsilon^3}\right)$.

We have that

$$\sum_{T \subseteq \mathcal{H}_3(G \setminus K(Q))} p_3^{|T|}(1-p_3)^{\binom{n}{3}-|T|} p_2^{-|K(T)|}$$

$$\geq \sum_{t=0}^{t_0} \sum_{\substack{T \subseteq \mathcal{H}_3(G \setminus Q) \\ |T|=t, I(T) \leq r}} p_3^t (1-p_3)^{\binom{n}{3}-t} p_2^{-6t+r}$$

$$\geq \sum_{t=0}^{t_0} (1-O(\varepsilon)) \binom{\binom{n}{3}}{t} p_2^{6t} \cdot p_3^t (1-p_3)^{\binom{n}{3}-t} p_2^{-6t+r}$$

$$\geq (1-O(\varepsilon)) p_2^r \cdot \sum_{t=0}^{t_0} p_3^t (1-p_3)^{\binom{n}{3}-t} \geq (1-O(\varepsilon)),$$

since $p_2^r = (1-e^{-mp^2(1-p)^{n-2}})^r \geq 1 - O(re^{-mp^2}) = 1 - O(\varepsilon)$.

Proof of Lemma 1

Lemma

1. $\mathrm{TV}\left(G(n,m;p), G(n,(p_k))\right) = o(1)$. (essentially by Rybarczyk)
2. $\mathrm{TV}(G(n,p_2,p_3,p_4), G(n,p_2)) = o(1)$. (Main part)
3. $\mathrm{TV}(G(n,p_2), G(n,\hat{p})) = o(1)$. (Not hard)

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Idea of Proof

1. Coupling argument
2. Property of Poisson distribution

Coupling argument

Definition

For two random variables X and Y, the coupling (X', Y') of X and Y is a random variable on the product of the sample spaces of X and Y such that the marginal distributions of X' and Y' are the distributions of X and Y, respectively.

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Definition

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Lemma

X, Y : random variables.

1. *Any coupling (X', Y') of X and Y satisfies*

$$\mathrm{TV}(X, Y) \leq \Pr[X' \neq Y'].$$

2. *There exists a coupling such that*

$$\mathrm{TV}(X, Y) = \Pr[X' \neq Y'].$$

Proof of Lemma (1)

- $X :=$ the number of columns of the matrix $R(n, m; p)$ with two or more 1's.
- $\Pr[|V_a| = k] = \binom{n}{k} p^k (1-p)^{n-k} =: r_k$
- $X = Binom(m, q_2)$ where $q_2 := \sum_{k \geq 2} r_k$.

$G(n, m; p)$ can be constructed as follows:

1. $K^{(1)}, \ldots, K^{(h)}, \ldots$: i.i.d. random complete graphs on subsets of V
   - the number of vertices in $K^{(1)}$ is $k(\geq 2)$ with probability r_k/q_2
   - Then, once the number is given to be k, every k-subset of V is equally likely to be the vertex set of $K^{(1)}$.
   - In other words, for a k-subset U of V with $k \geq 2$, the probability of U being the vertex set of $K^{(1)}$ is $\frac{r_k}{q_2}\binom{n}{k}^{-1}$.
2. $G(n, m; p)$ is the edge union of X random complete graphs $K^{(1)}, \ldots, K^{(X)}$.

Definition (G_Y)

- $Y := Poisson(mq_2)$ that is coupled with X so that

$$\Pr[X \neq Y] = \mathrm{TV}(X, Y).$$

- Let G_Y be the graph whose edge set is the (edge) union of $K^{(1)}, \ldots, K^{(Y)}$.

Property

1. G_Y has the same distribution as $G(n, (p_k))$.
2. $$\begin{aligned} \mathrm{TV}(G(n, m; p), G_Y) &\leq \Pr[G(n, m; p) \neq G_Y] \\ &\leq \Pr[X \neq Y] = \mathrm{TV}(X, Y). \end{aligned}$$

Lemma (Barbour and Holst (1989))

Let $X := Binom(m, q_2)$ and $Y := Poisson(mq_2)$. Then

$$\mathrm{TV}(X, Y) \leq q_2.$$

$$\mathrm{TV}(X, Y) \leq q_2 = \sum_{k \geq 2} \binom{n}{k} p^k (1-p)^{n-k} \leq \sum_{k \geq 2} n^k p^k = O\left(n^2 p^2\right)$$
$$= O\left(\frac{n^2 \log n}{m}\right) = o(1).$$

Problem

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Fix $3 < \alpha < 6$, and let $m = n^{\alpha}$.
Find a probability $p^* = p^*(n, m)$ such that

- If $p \ll p^*$, then $\mathrm{TV}\left(G(n, m; p), G(n, \hat{p})\right) = o(1)$.
- If $p \gg p^*$, then $\mathrm{TV}\left(G(n, m; p), G(n, \hat{p})\right) \geq c$, for some positive constant $c > 0$.

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Question

When $G(n, m; p) \not\approx G(n, \hat{p})$,
what are interesting properties and structures of $G(n, m; p)$?